Part V - Basket Options

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August, 2018

Our Hypothetical Problem

We have constructed a portfolio value equation where random portfolio value at time t is lognormally-distributed. The table below presents the portfolio parameters...

Table 1: Model Parameters

Symbol	Description	Value	Notes
$P(0)$	Portfolio value at time zero	1,000,000	In dollars
λ	Expected return mean	0.0423	Continuous-time rate
σ	Return volatility	0.1844	Continuous time-rate

Our task is to answer the following question:

Question: What is the value of a basket put option that expires at the end of year 3 given that the discrete-time risk-free rate is 3.00%, the discrete-time weighted average portfolio distribution rate is 4.10%, and the put option's exercise price is $800,000?

Porfolio Value Under Actual Probability Measure P

In Part IV of this series we defined the variable $P(t)$ to be portfolio value at time t, the variable λ to be expected return mean, the variable σ to be expected return volatility, and the variable Z to be a normally-distributed random variable with mean zero and variance one. The equation for random portfolio value at time t under the actual probability measure is...

$$P(t) = P(0)\,\mathrm{Exp}\left\{\lambda\,t + \sigma\sqrt{t}\,Z\right\} \text{ ...where... } Z \sim N\left[0,1\right] \tag{1}$$

Measure P is the actual probability distribution. Using Equation (1) above the equation for expected portfolio value at time t under Measure P is...

$$\mathbb{E}^P\left[P(t)\right] = \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}Z^2\right\} P(0)\,\mathrm{Exp}\left\{\lambda\,t + \sigma\sqrt{t}\,Z\right\}\delta Z \text{ ...where... } Z \sim N\left[0,1\right] \tag{2}$$

Given that random portfolio value via Equation (1) above is lognormally-distributed then the solution to expected portfolio value Equation (2) above is...

$$\mathbb{E}^P\left[P(t)\right] = P(0)\,\mathrm{Exp}\left\{\left(\lambda + \frac{1}{2}\sigma^2\right)t\right\} \tag{3}$$

Porfolio Value Under Risk-Neutral Probability Measure Q

We will define the variable α to be the continuous-time risk-free rate. Given that the discrete-time risk-free rate per our hypothetical problem is 3.00% the equation for the continuous-time risk-free rate is...

$$\alpha = \ln\left(1 + \text{risk-free rate}\right) = \ln\left(1 + 0.0300\right) = 0.0296 \tag{4}$$

We will define the variable ϕ to be the continuous time risk-free rate minus the continuous time distribution rate. Using the parameters from Table 1 above the equation for the variable ϕ is...

$$\phi = \ln\Bigg(1 + \text{risk-free rate} - \text{distribution rate}\Bigg) = \ln\Bigg(1 + 0.0300 - 0.0410\Bigg) = -0.0111 \tag{5}$$

Under the risk-neutral Measure Q all assets earn the risk-free rate. Using Equation (5) above the equation for expected portfolio value at time t under Measure Q is...

$$\mathbb{E}^Q\bigg[P(t)\bigg] = P(0)\,\text{Exp}\bigg\{\phi\,t\bigg\} \tag{6}$$

We will define the function $g(Z)$ to be the Girsanov multiplier given the normally-distributed random variable Z. This multiplier transforms Measure P (the actual probability distribution) to Measure Q (the risk-neutral probability distribution) by moving the mean but keeping the shape the same. The equation for the Girsanov multiplier for a standardized normal distribution is...

$$g(Z) = \text{Exp}\bigg\{\theta\,Z - \frac{1}{2}\,\theta^2\bigg\} \tag{7}$$

Using Equations (2) and (7) above the equation for expected portfolio value at time t under Measure Q is...

$$\begin{aligned}\mathbb{E}^Q\bigg[P(t)\bigg] &= \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\bigg\{-\frac{1}{2}\,Z^2\bigg\}\,P(0)\,\text{Exp}\bigg\{\lambda\,t + \sigma\sqrt{t}\,Z\bigg\}\,g(Z)\,\delta Z \\ &= P(0)\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\bigg\{-\frac{1}{2}\,Z^2\bigg\}\,\text{Exp}\bigg\{\lambda\,t + \sigma\sqrt{t}\,Z\bigg\}\,\text{Exp}\bigg\{\theta\,Z - \frac{1}{2}\,\theta^2\bigg\}\,\delta Z\end{aligned} \tag{8}$$

Using Appendix Equation (24) below the equation for the variable θ in the Girsanov Multiplier is...

$$\theta = \bigg(\phi - \lambda - \frac{1}{2}\,\sigma^2\bigg)t\bigg/\sigma\sqrt{t} \tag{9}$$

Basket Options

We will define the variable $B(t)$ to be the value of a basket put option at time t and the variable $X(t)$ to be the option exercise price at time t. The equation for the value of our basket put option at time zero is...

$$B(0) = \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\bigg\{-\frac{1}{2}\,Z^2\bigg\}\,\text{Max}\bigg(X(t) - P(t)\bigg)g(Z)\,\text{Exp}\bigg\{-\alpha\,t\bigg\}\,\delta Z \tag{10}$$

We will define the variable a to be the value of the normally-distributed random variable Z in Equation (10) such that portfolio value at time t equals the option exercise price at time t. Using Appendix Equation (25) below the value of variable a is...

$$a = \bigg[\ln\bigg(\frac{X(t)}{P(0)}\bigg) - \lambda\,t\bigg]\bigg/\sigma\sqrt{t} \tag{11}$$

Using Equation (11) above we can rewrite Equation (10) above as...

$$B(0) = \int\limits_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\bigg\{-\frac{1}{2}\,Z^2\bigg\}\bigg(X(t) - P(t)\bigg)g(Z)\,\text{Exp}\bigg\{-\alpha\,t\bigg\}\,\delta Z \tag{12}$$

Using Equations (7) and (12) above we will make the following definition...

$$\begin{aligned}I_1 &= \int\limits_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\bigg\{-\frac{1}{2}\,Z^2\bigg\}\,X(t)\,g(Z)\,\text{Exp}\bigg\{-\alpha\,t\bigg\}\,\delta Z \\ &= X(t)\,\text{Exp}\bigg\{-\alpha\,t\bigg\}\int\limits_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\bigg\{-\frac{1}{2}\,Z^2\bigg\}\,\text{Exp}\bigg\{\theta\,Z - \frac{1}{2}\,\theta^2\bigg\}\,\delta Z\end{aligned} \tag{13}$$

Using Equations (1), (7) and (12) above we will make the following definition...

$$
\begin{aligned}
I_2 &= \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\} P(t)\,g(Z)\,\mathrm{Exp}\left\{-\alpha\,t\right\}\delta Z \\
&= \int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\} P(0)\,\mathrm{Exp}\left\{\lambda\,t+\sigma\sqrt{t}\,Z\right\}\mathrm{Exp}\left\{\theta\,Z-\frac{1}{2}\,\theta^2\right\}\mathrm{Exp}\left\{-\alpha\,t\right\}\delta Z \\
&= P(0)\,\mathrm{Exp}\left\{\left(\lambda-\alpha\right)t\right\}\int\limits_{-\infty}^{a} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\}\mathrm{Exp}\left\{\sigma\sqrt{t}\,Z\right\}\mathrm{Exp}\left\{\theta\,Z-\frac{1}{2}\,\theta^2\right\}\delta Z
\end{aligned}
\tag{14}
$$

Using the definitions in Equations (13) and (14) above we can rewrite Equation (12) above as...

$$
B(0) = I_1 - I_2 \tag{15}
$$

Using Appendix Equations (30) and (36) below we can rewrite Equation (15) above as...

$$
\begin{aligned}
&B(0) = X(t)\,\mathrm{Exp}\left\{-\alpha\,t\right\}\mathrm{NORMSDIST}(d_1) - P(0)\,\mathrm{Exp}\left\{\left(\phi-\alpha\right)t\right\}\mathrm{NORMSDIST}(d_2) \\
&\text{where... } d_1 = \left[\ln\left(\frac{X(t)}{P(0)}\right) - \left(\phi-\frac{1}{2}\,\sigma^2\right)t\right]\Big/\sigma\sqrt{t} \;\text{ ...and... }\; d_2 = d_1 - \sigma\sqrt{t}
\end{aligned}
\tag{16}
$$

The Answer To Our Hypothetical Problem

Using Equations (5) (16) above and the parameters to our problem the value of the variable d_1 is...

$$
d_1 = \left[\ln\left(\frac{800,000}{1,000,000}\right) - \left(-0.0111 - \frac{1}{2}\times 0.1844^2\right)\times 3\right]\Big/ 0.1844\times\sqrt{3} = -0.4351 \tag{17}
$$

Using Equation (16) above and the parameters to our problem the value of the variable d_2 is...

$$
d_2 = -0.4351 - 0.1844\times\sqrt{3} = -0.7545 \tag{18}
$$

Appendix

A. If we equate Equations (6) and (8) above then the value of the variable θ in the Girsanov multiplier is...

$$
\begin{aligned}
P(0)\,\mathrm{Exp}\left\{\phi\,t\right\} &= P(0)\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\}\mathrm{Exp}\left\{\lambda\,t+\sigma\sqrt{t}\,Z\right\}\mathrm{Exp}\left\{\theta\,Z-\frac{1}{2}\,\theta^2\right\}\delta Z \\
\mathrm{Exp}\left\{\phi\,t\right\} &= \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\}\mathrm{Exp}\left\{\lambda\,t+\sigma\sqrt{t}\,Z+\theta\,Z-\frac{1}{2}\,\theta^2\right\}\delta Z \\
\mathrm{Exp}\left\{\phi\,t\right\} &= \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\left(Z^2-2\,\sigma\sqrt{t}\,Z-2\,\theta\,Z+\theta^2\right)\right\}\mathrm{Exp}\left\{\lambda\,t\right\}\delta Z \\
\mathrm{Exp}\left\{\left(\phi-\lambda\right)t\right\} &= \int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\left(Z^2-2\left(\sigma\sqrt{t}+\theta\right)Z+\theta^2\right)\right\}\delta Z
\end{aligned}
\tag{19}
$$

We will make the following definitions...

$$
Y = Z - \left(\sigma\sqrt{t}+\theta\right) \;\text{ ...where... }\; Y^2 = Z^2 - 2\left(\sigma\sqrt{t}+\theta\right)Z + \sigma^2 t + \theta^2 + 2\,\theta\,\sigma\sqrt{t} \tag{20}
$$

Using Equation (20) above we can rewrite Equation (19) above as...

$$\mathrm{Exp}\left\{\left(\phi-\lambda\right)t\right\}=\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Y^2\right\}\mathrm{Exp}\left\{\frac{1}{2}\,\sigma^2\,t+\theta\,\sigma\,\sqrt{t}\right\}\delta Z$$

$$\mathrm{Exp}\left\{\left(\phi-\lambda-\frac{1}{2}\,\sigma^2\right)t-\theta\,\sigma\,\sqrt{t}\right\}=\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Y^2\right\}\delta Z \tag{21}$$

Using Equation (20) above note the following...

$$\frac{\delta Y}{\delta Z}=1\ \text{ ...such that... }\ \delta Y=\delta Z \tag{22}$$

Using Equations (20) and (22) above we can rewrite Equation (21) above as...

$$\begin{aligned}\mathrm{Exp}\left\{\left(\phi-\lambda-\frac{1}{2}\,\sigma^2\right)t-\theta\,\sigma\,\sqrt{t}\right\}&=\int\limits_{-\infty-\sigma\sqrt{t}-\theta}^{\infty-\sigma\sqrt{t}-\theta}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Y^2\right\}\delta Y\\&=\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Y^2\right\}\delta Y\\&=1\end{aligned} \tag{23}$$

Taking the log of both sides of Equation (23) above and solving for the variable θ...

$$\left(\phi-\lambda-\frac{1}{2}\,\sigma^2\right)t-\theta\,\sigma\,\sqrt{t}=0\ \text{ ...such that... }\ \theta=\left(\phi-\lambda-\frac{1}{2}\,\sigma^2\right)t\Big/\sigma\sqrt{t} \tag{24}$$

B. Using Equation (2) above the value of the random variable Z where portfolio value at time t equals option exercise price at time t is...

$$\begin{aligned}P(t)&=X(t)\\P(0)\,\mathrm{Exp}\left\{\lambda\,t+\sigma\sqrt{t}\,Z\right\}&=X(t)\\\lambda\,t+\sigma\sqrt{t}\,Z&=\ln\left(\frac{X(t)}{P(0)}\right)\\Z&=\left[\ln\left(\frac{X(t)}{P(0)}\right)-\lambda\,t\right]\Big/\sigma\sqrt{t}\end{aligned} \tag{25}$$

C. The solution to Equation (13) above is...

$$\begin{aligned}I_1&=X(t)\,\mathrm{Exp}\left\{-\alpha\,t\right\}\int\limits_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\}\mathrm{Exp}\left\{\theta\,Z-\frac{1}{2}\,\theta^2\right\}\delta Z\\&=X(t)\,\mathrm{Exp}\left\{-\alpha\,t\right\}\int\limits_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\left(Z^2-2\,\theta\,Z+\theta^2\right)\right\}\delta Z\end{aligned} \tag{26}$$

We will make the following definition...

$$Y=Z-\theta\ \text{ ...where... }\ Y^2=Z^2-2\,\theta Z+\theta^2\ \text{ ...and... }\ \frac{\delta Y}{\delta Z}=1\ \text{ ...and... }\ \delta Y=\delta Z \tag{27}$$

Using Equation (27) above we can rewrite Equation (26) above as...

$$\begin{aligned}
I_1 &= X(t)\,\mathrm{Exp}\bigg\{-\alpha\,t\bigg\}\int\limits_{-\infty-\theta}^{a-\theta}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\bigg\{-\frac{1}{2}\,Y^2\bigg\}\,\delta Y\\
&= X(t)\,\mathrm{Exp}\bigg\{-\alpha\,t\bigg\}\int\limits_{-\infty}^{a-\theta}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\bigg\{-\frac{1}{2}\,Y^2\bigg\}\,\delta Y\\
&= X(t)\,\mathrm{Exp}\bigg\{-\alpha\,t\bigg\}\,\mathrm{NORMSDIST}(a-\theta)
\end{aligned}\tag{28}$$

Note that NORMSDIST(value) is an Excel function.

Using Equations (9) and (11) above we will define the variable d_2 as follows...

$$\begin{aligned}
d_1 &= a-\theta\\
&= \left[\ln\left(\frac{X(t)}{P(0)}\right)-\lambda\,t\right]\bigg/\sigma\sqrt{t}-\left(\phi-\lambda-\frac{1}{2}\,\sigma^2\right)t\bigg/\sigma\sqrt{t}\\
&= \left[\ln\left(\frac{X(t)}{P(0)}\right)-\lambda\,t-\phi\,t+\lambda\,t+\frac{1}{2}\,\sigma^2 t\right]\bigg/\sigma\sqrt{t}\\
&= \left[\ln\left(\frac{X(t)}{P(0)}\right)-\left(\phi-\frac{1}{2}\,\sigma^2\right)t\right]\bigg/\sigma\sqrt{t}
\end{aligned}\tag{29}$$

Using Equation (29) above we can rewrite Equation (28) above as...

$$I_1 = X(t)\,\mathrm{Exp}\bigg\{-\alpha\,t\bigg\}\,\mathrm{NORMSDIST}(d_1)\ \text{ ...where... }\ d_1 = \left[\ln\left(\frac{X(t)}{P(0)}\right)-\left(\phi-\frac{1}{2}\,\sigma^2\right)t\right]\bigg/\sigma\sqrt{t}\tag{30}$$

D. The solution to Equation (14) above is...

$$\begin{aligned}
I_2 &= P(0)\,\mathrm{Exp}\bigg\{\Big(\lambda-\alpha\Big)t\bigg\}\int\limits_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\bigg\{-\frac{1}{2}\,Z^2\bigg\}\,\mathrm{Exp}\bigg\{\sigma\sqrt{t}\,Z\bigg\}\,\mathrm{Exp}\bigg\{\theta\,Z-\frac{1}{2}\,\theta^2\bigg\}\,\delta Z\\
&= P(0)\,\mathrm{Exp}\bigg\{\Big(\lambda-\alpha\Big)t\bigg\}\int\limits_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\bigg\{-\frac{1}{2}\Big(Z^2-2\Big(\sigma\sqrt{t}+\theta\Big)Z+\theta^2\Big)\bigg\}\,\delta Z
\end{aligned}\tag{31}$$

We will make the following definition...

$$Y = Z-\Big(\sigma\sqrt{t}+\theta\Big)\ \text{ ...where... }\ Y^2 = Z^2-2\Big(\sigma\sqrt{t}+\theta\Big)Z+\sigma^2 t+2\,\theta\sigma\sqrt{t}+\theta^2\ \text{ ...and... }\ \frac{\delta Y}{\delta Z}=1\ \text{ ...and... }\ \delta Y=\delta Z\tag{32}$$

Using Equation (32) above we can rewrite Equation (31) above as...

$$\begin{aligned}
I_2 &= P(0)\,\mathrm{Exp}\bigg\{\Big(\lambda-\alpha\Big)t\bigg\}\int\limits_{-\infty-\sigma\sqrt{t}-\theta}^{a-\sigma\sqrt{t}-\theta}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\bigg\{-\frac{1}{2}\Big(Y^2-\sigma^2 t-2\,\theta\sigma\sqrt{t}\Big)\bigg\}\,\delta Y\\
&= P(0)\,\mathrm{Exp}\bigg\{\Big(\lambda-\alpha\Big)t\bigg\}\int\limits_{-\infty}^{a-\sigma\sqrt{t}-\theta}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\bigg\{-\frac{1}{2}\,Y^2\bigg\}\,\mathrm{Exp}\bigg\{\frac{1}{2}\,\sigma^2 t+\theta\sigma\sqrt{t}\bigg\}\,\delta Y\\
&= P(0)\,\mathrm{Exp}\bigg\{\Big(\lambda-\alpha+\frac{1}{2}\,\sigma^2\Big)t+\theta\,\sigma\sqrt{t}\bigg\}\int\limits_{-\infty}^{a-\sigma\sqrt{t}-\theta}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\bigg\{-\frac{1}{2}\,Y^2\bigg\}\,\delta Y\\
&= P(0)\,\mathrm{Exp}\bigg\{\Big(\lambda-\alpha+\frac{1}{2}\,\sigma^2\Big)t+\theta\,\sigma\sqrt{t}\bigg\}\,\mathrm{NORMSDIST}(a-\sigma\sqrt{t}-\theta)
\end{aligned}\tag{33}$$

Using Equations (9) and (33) above note the following...

$$\left(\lambda-\alpha+\frac{1}{2}\,\sigma^2\right)t+\theta\,\sigma\sqrt{t}=\left(\lambda-\alpha+\frac{1}{2}\,\sigma^2\right)t+\left(\phi-\lambda-\frac{1}{2}\,\sigma^2\right)t\Big/\sigma\sqrt{t}\Big)\sigma\sqrt{t}$$
$$=\left(\lambda-\alpha+\frac{1}{2}\,\sigma^2+\phi-\lambda-\frac{1}{2}\,\sigma^2\right)t$$
$$=\Big(\phi-\alpha\Big)t \tag{34}$$

Using Equations (29) above we will define the variable d_1 as follows...

$$d_2=a-\sigma\sqrt{t}-\theta$$
$$=d_1-\sigma\sqrt{t} \tag{35}$$

Using Equations (34) and (35) above we can rewrite Equation (33) above as...

$$I_2=P(0)\,\mathrm{Exp}\left\{\Big(\phi-\alpha\Big)t\right\}\mathrm{NORMSDIST}(d_2)\ \text{ ...where... }\ d_2=d_1-\sigma\sqrt{t} \tag{36}$$